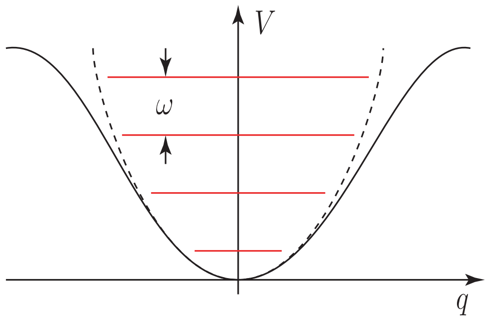
Lecture XII: Applications of the Feynman Path Integral

▷ Digression: Free particle propagator (exercise) cf. diffusion

$$G_{\rm free}(q_F, q_I; t) \equiv \langle q_F| e^{-i\hat{p}^2 t/2m\hbar}|q_I\rangle \Theta(t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left[\frac{i}{\hbar}\frac{m(q_F-q_I)^2}{2t}\right]\Theta(t)$$

Difficult to derive from PI(!), but useful for normalization

▷ Quantum Particle in a Single (Symmetric) Well: $V(q) = V(-q)$



e.g. QM amplitude

$$G(0,0;t) \equiv \langle 0|e^{-i\hat{H}t/\hbar}|0\rangle\, \Theta(t) = \int_{q(t)=q(0)=0} Dq \exp\left[\frac{i}{\hbar}\int_0^t dt'\left(\frac{m\dot{q}^2}{2} - V(q)\right)\right]$$

▷ Evaluate PI by stationary phase approximation: *general recipe*

(i) Parameterise path as $q(t) = q_{\rm cl}(t) + r(t)$ and expand action in $r(t)$

$$S[\bar{q}+r] = \int_0^t dt'\left[\frac{m}{2}\overbrace{(\dot{q}_{\rm cl}+\dot{r})^2}^{\dot{q}_{\rm cl}^2 + 2\dot{q}_{\rm cl}\dot{r} + \dot{r}^2} - \overbrace{V(q_{\rm cl}+r)}^{V(q_{\rm cl}) + rV'(q_{\rm cl}) + \frac{r^2}{2}V''(q_{\rm cl}) + \cdots}\right]$$

$$= S[q_{\rm cl}] + \int_0^t dt' r(t')\overbrace{[-m\ddot{q}_{\rm cl} - V'(q_{\rm cl})]}^{\frac{\delta S}{\delta q(t')}=0} + \frac{1}{2}\int_0^t dt' r(t')\overbrace{[-m\partial_{t'}^2 - V''(q_{\rm cl})]}^{\frac{\delta^2 S}{\delta q(t')\delta q(t'')}} r(t') + \cdots$$

(ii) Classical trajectory: $m\ddot{q}_{\rm cl} = -V'(q_{\rm cl})$

Many solutions — choose non-singular solution $q_{\rm cl} = 0$ *(why?)*

i.e. $S[q_{\rm cl}] = 0$ and $V''(q_{\rm cl}) = m\omega^2$ constant

$$G(0,0;t) \simeq \int_{r(0)=r(t)=0} Dr \exp\left[\frac{i}{\hbar}\int_0^t dt' r(t')\frac{m}{2}\left(-\partial_{t'}^2 - \omega^2\right) r(t')\right]$$

N.B. if V was quadratic, expression trivially exact

More generally, $q_{\rm cl}(t)$ non-trivial $\mapsto$ non-vanishing $S[q_{\rm cl}]$ — see PS3

Fluctuation contribution? — example of a...

▷ Gaussian functional integration: *mathematical interlude*

- One variable Gaussian integral: $(\int_{-\infty}^{\infty} dv\, e^{-av^2/2})^2 = 2\pi \int_0^{\infty} r\, dr\, e^{-ar^2/2} = \frac{2\pi}{a}$

$$\int_{-\infty}^{\infty} dv\, e^{-\frac{a}{2}v^2} = \sqrt{\frac{2\pi}{a}}, \qquad \mathrm{Re}\, a > 0$$

- More than one variable:

$$\int d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} = (2\pi)^{N/2}\det \mathbf{A}^{-1/2}$$

where $\mathbf{A}$ is +ve definite real symmetric $N \times N$ matrix

Proof: $\mathbf{A}$ diagonalised by orthogonal transformation: $\mathbf{A} = \mathbf{O}^T\mathbf{D}\mathbf{O}$

Change of variables: $\mathbf{w} = \mathbf{O}\mathbf{v}$ (Jacobian $\det(\mathbf{O}) = 1$) $\rightsquigarrow$ N decoupled Gaussian integrations: $\mathbf{v}^T\mathbf{A}\mathbf{v} = \mathbf{v}^T\mathbf{O}^T\mathbf{O}\mathbf{A}\mathbf{O}^T\mathbf{O}\mathbf{v} = \mathbf{w}^T\mathbf{D}\mathbf{w} = \sum_i^N d_i w_i^2$

Finally, $\prod_{i=1}^N d_i = \det \mathbf{D} = \det \mathbf{A}$

- Infinite number of variables; interpret $\{v_i\} \mapsto v(t)$ as continuous field and $A_{ij} \mapsto A(t,t') = \langle t|\hat{A}|t'\rangle$ as operator kernel

$$\int Dv(t)\, \exp\left[-\frac{1}{2}\int dt \int dt'\, v(t)A(t,t')v(t')\right] \propto (\det \hat{A})^{-1/2}$$

(iii) Applied to quantum well, $A(t,t') = -\frac{i}{\hbar}m\delta(t-t')(-\partial_{t'}^2 - \omega^2)$ and formally

$$G(0,0;t) \simeq J \det\left(-\partial_{t'}^2 - \omega^2\right)^{-1/2}$$

where J absorbs various constant prefactors (im, $\hbar$, etc.)

What does 'det' mean? Effectively, we have expanded trajectories $r(t')$ in eigenbasis of $\hat{A}$ subject to b.c. $r(t) = r(0) = 0$

$$\left(-\partial_t^2 - \omega^2\right) r_n(t) = \epsilon_n r_n(t), \qquad \text{cf. PIB}$$

i.e. Fourier series expn: $r_n(t') = \sin(\frac{n\pi t'}{t})$, $\quad n = 1, 2, \ldots, \qquad \epsilon_n = (\frac{n\pi}{t})^2 - \omega^2$

$$\det\left(-\partial_t^2 - \omega^2\right)^{-1/2} = \prod_{n=1}^{\infty} \epsilon_n^{-1/2} = \prod_{n=1}^{\infty}\left(\left(\frac{n\pi}{t}\right)^2 - \omega^2\right)^{-1/2}$$

▷ For $V = 0$, $G = G_{\text{free}}$ known — use to eliminate constant prefactor J

$$G(0,0;t) = \frac{G(0,0;t)}{G_{\text{free}}(0,0;t)} G_{\text{free}}(0,0;t) = \prod_{n=1}^{\infty}\left[1 - \left(\frac{\omega t}{n\pi}\right)^2\right]^{-1/2}\left(\frac{m}{2\pi i\hbar t}\right)^{1/2}\Theta(t)$$

Finally, applying identity $\prod_{n=1}^{\infty}[1 - (\frac{x}{n\pi})^2]^{-1} = \frac{x}{\sin x}$

$$G(0,0;t) \simeq \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}}\Theta(t)$$

(exact for harmonic oscillator)